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Optimizing Asset and Capital Adequacy Management in Banking

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Abstract This paper considers the application of stochastic optimization theory to asset and capital adequacy management in banking. Our study is motivated by new banking regulation that emphasizes risk minimization practices associated with assets and regulatory capital. Our analysis depends on the dynamics of the capital adequacy ratio (CAR), which we compute in a stochastic setting, by dividing regulatory bank capital (RBC) by risk weighted assets (RWAs). Furthermore, we demonstrate how the CAR can be optimized in terms of bank equity allocation and the rate at which additional debt and equity is raised. In either case, the dynamic programming algorithm for stochastic optimization is employed to verify the results. Also, we provide an illustration of aspects of bank management practice in relation to this regulation. Finally, we make a few concluding remarks and discuss possibilities for further research.

Keywords Bank management · Assets · Capital adequacy · Stochastic optimization

1 Introduction

Bank asset management mainly involves achieving profit maximization via high returns on loans and securities, reducing risk and providing for liquidity needs. More specifically, banks try to manage their assets in the following ways. They endeavor

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to grant loans to creditors who are likely to pay high interest rates and are unlikely to default on their loans. Secondly, banks try to purchase securities with high returns and low risk. Also, in managing their assets, banks attempt to lower risk by diversifying their investment portfolio. The study of the dynamics of these risk minimization strategies has always been an important issue in the management of banks. In particular, [1, 2] construct continuous-time models which permit optimization problems to be solved in the context of portfolio selection and capital requirements. Finally, the bank must manage the liquidity of its assets in order to satisfy possible reserve requirements (compare capital requirements à la Basel II) without incurring high costs. On the other hand, capital adequacy management involves the decision about the amount of capital the bank should hold and how it should be accessed [3]. Bank capital management bears a double burden since capital benefits the bank owners because it reduces the likelihood of bank failure while being costly because the higher the level of capital the lower will be the return on equity for a prescribed return on assets. Thus, when determining the amount of capital to hold, the bank owner must decide on how much of the increased benefit that results from the higher capital they are willing to trade-off against the lower return on equity that originates from the cost associated with higher capital.

An important factor influencing asset and capital adequacy management is regulation and supervision. The high cost of capital provides an incentive for bank owners to retain less capital relative to assets than is required by regulatory authorities. In this situation, the amount of bank capital to hold is prescribed by certain capital requirements. The first of these is based on the *leverage ratio* that is calculated by dividing the regulatory capital by the bank's total assets. An agreement among banks from industrialized countries brought the *Basel Committee on Banking Supervision* (BCBS), that meets under the auspices of the Banks for International Settlements (BIS) in Basel, Switzerland, into being. To date the BCBS has implemented capital regulation in the form of the so-called *Basel Accords* ([4] and its amendments [5–7] for the Basel I Capital Accord; [8–10] for the Basel II Capital Accord), where a second type of capital requirement that is risk-based is considered. Measures of capital adequacy are generally calculated using the book values of assets and equity. Greater levels of regulation generally entail additional costs for the bank. Currently, this regulation takes the form of the Basel II Capital Accord [8, 10] that is to be implemented on a worldwide basis by 2007. Basel II is based on three pillars related to minimum capital requirements, supervisory review and market discipline ([11] for a discussion on the interaction between these pillars).

The 1996 Amendment's Internal Models Approach (IMA) determines the capital requirements on the basis of the banks' internal risk measurement systems. Banks are required to report daily their value-at-risk (VaR) at a 99% confidence level over both a one day and two weeks (10 trading days) horizon. The minimum capital requirement is then the sum of a premium to cover *credit risk* [12], a premium to cover general *market risk* [5, 6] and a premium to cover *operational risk* [13]. The *credit risk premium* is made up of 8% of the risk-weighted assets and the *market risk premium* is equal to a multiple of the average reported two-week VaRs in the last 60 trading days. The *operational risk premium* is calculated by considering the risk associated with each of eight business lines. The impact of a risk-sensitive framework such as Basel II on macroeconomic stability of banks is an important issue. In order for a bank to determine their minimum capital requirements they will first decide on a planning horizon. This planning horizon is then divided into non-overlapping backtesting-periods, which is in turn divided into non-overlapping reporting periods. At the start of each reporting period the bank has to report its VaR for the current period and the actual loss from the previous period. The market risk premium for the current reporting period is then equal to the multiple m of the reported VaR. At the end of each backtesting period, the number of reporting periods in which actual loss exceeded VaR is counted and this determines the multiple m for the next backtesting period according to a given increasing scale.

1.1 Relation to Previous Literature

In this subsection, we consider the association between our contribution and previous literature. The issues that we highlight include the role of bank capital, capital regulation and stochastic modeling and optimization.

The most important role of capital is to mitigate the moral hazard problem that results from asymmetric information between banks, depositors and borrowers. The Modigliani–Miller theorem forms the basis for modern thinking on capital structure [14]. In an efficient market, their basic result states that, in the absence of taxes, insolvency costs and asymmetric information, the bank value is unaffected by how it is financed. In this framework, it does not matter if bank capital is raised by issuing equity or selling debt or what the dividend policy is. By contrast, in our contribution, in the presence of loan market frictions, the bank value is dependent on its financial structure [3, 15–17]. Furthermore, in the presence of asymmetric information about loans, bank owners may be aware of asset quality problems unknown to outside analysts. Provisioning the assets may convey a clearer picture regarding the worth of these assets and precipitate a (negative) market adjustment. In this case, it is well known that the bank's decisions about lending and other issues may be driven by the CAR [2, 11, 18–20].

Banks are among the most heavily regulated of all financial institutions. In particular, capital requirements have become important components of regulation and supervision in the banking industry. As from June 1999, the Basel Committee on Banking Supervision (BCBS) released several proposals [5–7] to reform the original 1988 Basel Capital Accord [4]. These efforts culminated in the Basel II Capital Accord [8–10]. Because our aim is to study capital requirements in regulatory policy, we need to consider dynamic models of bank behavior, since static models are not able to capture the effects of such requirements ([21] for more details). Several studies related to optimization problems in banking have recently surfaced in the literature [16, 20, 22]. Also, other papers that use dynamic optimization methods to analyze bank regulatory capital policies include [23] for Basel II and [24, 25] for Basel market risk capital requirements. In [20], a discrete-time dynamic banking model of imperfect competition is presented, where the bank can invest in a prudent or a gambling asset. For both these options, a maximization problem that involves the bank value for shareholders is formulated. On the other hand, [22] examines a problem related to the optimal risk management of banks in a continuous-time stochastic dynamic setting.

In particular, the authors minimize market and capital adequacy risk that involves the safety of the assets held and the stability of sources of capital, respectively [26–28].

1.2 Preliminaries

At the outset, we assume that the stochastic process, θ, represents investment in the bank's risky assets. The dynamics of the value of the bank's asset portfolio A, over any reporting period, for $A(t_0) = A_0$, may then be given by

$$dA(t) = \big(A(t)r^{\mathrm{T}}(t) + \theta(t)^T \mu\big)dt + \theta(t)^T \sigma^A(t)dZ^A(t) - r^{\mathrm{T}}(t)D(t)dt,$$

where μ is an appropriate drift parameter, r^{T} is the Treasuries rate and the last term reflects interest paid to depositors with D denoting the deposits. In this case, we define the bank's regulatory capital K as

$$K = A - D.$$

The bank is required to maintain this capital above a minimum level equal to the sum of the charge to cover general market risk plus a charge to cover credit (or idiosyncratic) risk plus a charge to cover operational risk [13]. The charge to cover operational risk equals the sum of the charges for each of eight business lines (corporate finance, trading and sales, retail banking, commercial banking, payment and settlement, agency services, asset management and retail brokerage). More specifically, the capital charge for operational risk, under the Standardized Approach outlined in Basel II, may be expressed as

$$\max\left[\sum_{k=1}^{8} \beta_k g_k, 0\right],$$

where

g_{1-8}: Three-Year Average of Gross Income for Eight Business Lines;
β_{1-8}: Fixed Percentage Relating Level of Required Capital to Level of Gross Income for Each of Eight Business Lines.

The β-values for operational risk are provided in the document [13]. The charge to cover market risk equals the VaR reported at the beginning of the current reporting period times a multiple m. The charge to cover credit risk equals the sum of the bank's trading positions (long and short) multiplied by asset-specific risk weights. In the sequel, we assume that the RWAs are expressible as

$$A^r(t) = \omega^A \Lambda(t) + \omega^I I(t),$$

where ω^Λ and ω^I are the Basel II risk weights for the loans Λ and intangible assets I, respectively. As a consequence of the above, if $VaR \geq 0$ denotes the VaR reported to regulators at the beginning of the current reporting period and m is the currently-applicable multiple, the bank must satisfy the constraint

$$K(t) \geq mVaR + \omega^\Lambda \Lambda(t) + \omega^I I(t) + \max\left[\sum_{k=1}^{8} \beta_k g_k, 0\right],$$

at all times during the reporting period. The reported VaR can differ from the true VaR since the bank's future trading strategy, and hence the bank's true VaR, are unobservable by regulators. Despite this, our study only considers a simplified version of the capital adequacy ratio (CAR) of the form

$$\mathrm{CAR}(\kappa) = \frac{\mathrm{RBC}(K)}{\mathrm{RWAs}(A^r)}. \tag{1}$$

In other words, in the calculation of the CAR, the total risk charge (TRC) is only constituted by the risk weighted assets (RWAs). Essentially, banks strive to maintain κ in excess of some CAR regulatory benchmark ρ with supervisory intervention resulting if this is not the case. Despite the fact that more than 100 countries will be Basel II-compliant by the end of the year 2007, limitations in this regulatory framework have become apparent [19, 22, 29, 30]. For instance, Basel II gives a precise description of the bank capital and TRCs to be used in the computation of κ in (1), but neglects to provide complete details of reference processes and thresholds for bank closure, shirking, corrective action and continuance in relation to κ.

1.3 Highlights of the Paper

In this paper, our primary objective is to model the main measure of capital adequacy, namely the capital adequacy ratio. In order to compute this ratio, we consider the stochastic dynamics of items such as on- and off-balance sheet assets, liabilities, regulatory capital and CARs in a Lévy process setting. We summarize the main highlights of our paper below.

The first highlight of our work is related to the construction of a stochastic dynamic model for risk-based CARs (Proposition 2.1 in Sect. 2) according to some of the main prescripts of the Basel II Capital Accord. A stochastic optimization problem in banking is stated in Problem 3.1 of Sect. 3.1 and solved in Theorem 3.1 of Sect. 3.2. Thereafter, a comprehensive discussion of the cost function is provided in Sect. 5.2.1. In this regard, the main aims when choosing the cost are to meet the bank's regulatory obligations and to elicit as little additional debt and equity from the debt- and shareholders, respectively, as possible. While the reason for the first objective is obvious, a partial economic motivation for the latter is that the cost of raising capital is extremely high. In Theorem 3.2 in Sect. 3.2, we determine the level of returns on bank equity and rate at which additional debt and equity is raised that is needed to attain an optimal CAR level via a quadratic cost function. An indepth analysis of the control law is subsequently done in Sect. 5.2.2. In Sect. 4, we illustrate how our stochastic dynamic models for bank behavior are able to facilitate an improvement in risk management and regulation in practice. The economic importance of our illustration is that it has a strong correlation with the 3 pillars of the Basel II Accord, viz., minimum capital requirement, supervisory review and market discipline. In this part of the paper, we firstly consider the probability of default of granted loans. The second feature of our example involves the implications of and the interactions between the three pillars of Basel II regulation for bank management that includes a consideration of a regulatory capital constraint. In this regard, we highlight the dynamic interaction between a regulator and bank owner. Here, information provided by the bank owner about the level of bank capital is important for the decision by the supervisor on whether to allow the bank to continue to function or enforce bank closure. Finally, throughout the example, risk incentives, risk shifting and other constraints on banking behavior are referred to. With regard to the latter, realistic constraints associated with the eliciting of additional debt and equity, profitability, incentive compatibility and financing are brought to bear on bank management practice. We provide a discussion on one of the perceived shortcomings of Basel II regulation by introducing the notion of a CAR reference process (see Sect. 5.2.2 for the analysis of such processes).

2 Banking Model

In this section, we show that concepts related to banking such as risky asset losses, returns on bank equity and the RWA level may be modeled as random variables that are driven by an associated Brownian motion or Wiener process. In this regard, the dynamics of each of these banking items is expressed explicitly in the form of a stochastic differential equation (SDE). Throughout we assume that we are working with a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ on a reporting period $T = [t_0, t_1]$.

2.1 Regulatory Capital and RWAs

The total bank capital K can be expressed as

$$K(t) = K^{T1}(t) + K^{T2}(t) + K^{T3}(t),$$

where $K^{T1}(t)$, $K^{T2}(t)$ and $K^{T3}(t)$ are Tier 1, Tier 2 and Tier 3 capital, respectively. *Tier 1* (T1) capital is the book value of the bank's equity, y, plus retained earnings E_r. Tier 2 (T2) and Tier 3 (T3) capital (collectively known as *supplementary capital*) is the sum of subordinate debt, y_0, and loan-loss reserves R_L. Because of their non-dynamic nature, in the sequel, we do not consider E_r and R_L to be active constituents of total capital, so that

$$dK^{T1}(t) = dy(t) \text{ and } dK^{T2}(t) + dK^{T3}(t) = dy_0(t).$$

In this subsection, we are able to produce a system of SDEs that provide information about the *total bank capital* at time t with $K : \Omega \times T \to \mathbb{R}$ denoted by $K(t)$ and RWAs at time t with $A^r : \Omega \times T \to \mathbb{R}$ denoted by $A^r(t)$.

2.1.1 Description of the Regulatory Capital and RWAs

Bank capital is raised by selling new equity, retaining earnings, issuing debt or building up loan-loss reserves. Responsibility for calculating capital adequacy requirements is usually borne by the bank's Risk Management Department. Calculated risk capital is then approved by a bank's top executive management. Furthermore, the structure of capital (i.e., Tier 1, Tier 2 and Tier 3) are proposed by the Finance Department and subsequently approved by the bank's top executive management. The dynamics of bank capital K is stochastic in nature because it depends in part on the uncertainty related to debt- and shareholder contributions. In theory, the bank can decide on the rate at which debt and equity is raised. The underlying principle governing this decision is that the level of capitalization of the bank has to be taken into account. Roughly speaking, the rate at which debt and equity is raised can be reduced during times when the bank is adequately capitalized and should be increased beyond the normal rate when the bank is undercapitalized. When using the Basel II risk-based approach to assets, RWAs are defined by placing each on- and off-balance item into a risk category with a prescribed risk weight. In this regard, the riskier the asset the higher the risk-weight. In our case, on- and off-balance sheet assets are allocated to five categories each with a different weight. The first category carries a 0% weight and includes items that have little default risk, such as reserves and government securities in Organization for Economic Co-operation and Development (OECD) countries. Category 2 has a 20% weight and includes claims on banks in OECD countries. The third category carries a weight of 50% and includes municipal bonds and residential mortgages. Category 4 has the maximum weight of 100% and includes loans to consumers and corporations. Off-balance sheet items form the fifth category and are treated in a similar manner by assigning a credit-equivalent percentage that converts them to on-balance sheet items to which the appropriate risk weight applies. The main constituents of this category are intangible assets that carry a risk weight of 100% and are used in determining the value of Tier 1 capital.

2.1.2 Mathematical Relationships between Regulatory Capital and RWAs

In the sequel, the stochastic process $u_1 : \Omega \times T \to \mathbb{R}$ is the *normal rate at which debt and equity is raised per monetary unit of RWAs* whose value at time t is denoted by $u_1(t)$. A notion related to this is the *adjustment to the rate at which debt and equity is raised per monetary unit of RWAs for over- or undercapitalization*, $u_2 : \Omega \times T \to \mathbb{R}$, whose value at time t is denoted by $u_2(t)$. In closed loop u_2 will be dependent on the CAR. Here the amount of over- or undercapitalization is reliant on the excess of bank capital over RWAs. We denote the sum total of u_1 and u_2 by the rate $u_3 : \Omega \times T \to \mathbb{R}$, where

$$u_3(t) = u_1(t) + u_2(t), \quad \text{for all } t. \tag{2}$$

The rate at which total debt and equity is raised, u_3, is assumed to be a predictable process and, as we shall see in the sequel, provides us with a means of controlling the CAR dynamics of the bank. The closed loop system will be defined such that this assumption is met. With regard to (2), we could possibly consider choosing the rate at which additional debt and equity is raised, u_2, sufficiently large so that the solvency of the bank is guaranteed. However, as was mentioned before, a decrease in returns on equity that is commensurate with the cost of holding more bank capital will result. The *risky asset losses per monetary unit of RWAs*, $l : \Omega \times T \to \mathbb{R}$, whose value at time t is denoted by $l(t)$, is given by

$$dl(t) = r_l(t)dt + \sigma_l(t)dZ_l(t), \qquad l(t_0) = l_0, \tag{3}$$

where $r_l : T \to \mathbb{R}$ is the rate of asset loss per monetary unit of RWAs, $l(t)$ is a random variable, $\sigma_l : T \to \mathbb{R}$ is the volatility in the risky asset losses per monetary unit of

RWAs and $Z_l : \Omega \times T \to \mathbb{R}$ is a standard Brownian motion whose value at time t is denoted by $Z_l(t)$. As far as the bank capital is concerned, we consider

$$dh(t) = r_h(t)dt + \sigma_h(t)dZ_h(t), \qquad h(t_0) = h_0, \tag{4}$$

where the random variable $h : \Omega \times T \to \mathbb{R}$ in (4), whose value at time t is denoted by $h(t)$, $r_h : T \to \mathbb{R}$ is the *rate of return on capital per monetary unit of the bank's capital*, $\sigma_h : T \to \mathbb{R}$, is the volatility in the rate of return on bank capital and $Z_h : \Omega \times T \to \mathbb{R}$ is a standard Brownian motion whose value at time t is denoted by $Z_h(t)$. We suppose from the outset that the debt- and shareholders invest in a bank with $n+1$ opportunities for investment in the debt and equity components of bank capital. One of these bank capital categories is risk free and corresponds to *subordinate debt*. Categories $1, 2, \dots, n$ are risky and is constituted by different classes of *bank equity*. These categories of bank capital evolve continuously in time and are modeled using a n-dimensional Brownian motion. In this multidimensional context, the *return on bank capital in the k-th capital class per monetary unit of the k-th capital class* is denoted by $y_k(t)$, $k \in \mathbb{N}_n = \{0, 1, 2, \dots, n\}$ where $y : \Omega \times T \to \mathbb{R}^{n+1}$. We can represent y as $y = (y_0(t), y_1(t), \dots, y_n(t))$, where $y_0(t)$ is subordinate debt and $y_1(t), \dots, y_n(t)$ are bank equities. Furthermore, we can model y as

$$dy(t) = r_y(t)dt + \Sigma_y(t)dZ_y(t), \qquad y(t_0) = y_0, \tag{5}$$

with $r_y : T \to \mathbb{R}^{n+1}$, $Z_y : \Omega \times T \to \mathbb{R}^n$ and $\Sigma_y(t) \in \mathbb{R}^{(n+1)\times n}$, where there are only n scalar Brownian motions due to the subordinate debt being riskless. Denote the proportion invested in the bank capital by $\pi(t) = (\pi_0(t), \pi_1(t), \dots, \pi_n(t))^T$, $\pi : T \to \mathbb{R}^{n+1}$. The *return on bank capital* is then $h : \Omega \times R \to \mathbb{R}$,

$$dh(t) = \pi(t)^T dy(t) = \pi(t)^T r_y(t)dt + \pi(t)^T \Sigma_y(t)dZ_y(t).$$

Due to the fact that the proportions of the total bank capital, the components of the vector $\pi(t)$, sum to 1 for all $t \in T$, the notation can be simplified. In the sequel, the $\tilde{\cdot}$ notation refers to the notion $\cdot$ that can be associated with the risky (equity) component of the bank capital. In this spirit, denote

$$\begin{aligned}
&r_0(t) = r_{y,0}(t), \quad r_0 : T \to \mathbb{R}, \text{ rate of return on subordinate debt,}\\
&r_y(t) = (r_0(t), \tilde{r}_y(t)^T + r_0(t)1_n)^T, \quad \tilde{r}_y : T \to \mathbb{R}^n,\\
&\pi(t) = (\pi_0(t), \pi_1(t), \dots, \pi_k(t))^T, \quad \tilde{\pi} : T \to \mathbb{R}^k,\\
&\Sigma_y(t) = \begin{pmatrix} 0 & \dots & 0 \\ & \tilde{\Sigma}_y(t) & \end{pmatrix}, \quad \tilde{\Sigma}_y(t) \in \mathbb{R}^{n\times n}, \qquad \tilde{C}(t) = \tilde{\Sigma}_y(t)\tilde{\Sigma}_y(t)^T > 0.
\end{aligned}$$

Then,

$$\begin{aligned}
&\pi(t)^T r_y(t) = \pi_0(t) r_0(t) + \tilde{\pi}_j(t)^T \tilde{r}_y(t) + \tilde{\pi}_j(t)^T r_0(t)1_n = r_0(t) + \tilde{\pi}(t)^T \tilde{r}_y(t),\\
&\pi(t)^T \Sigma_y(t)dZ_y(t) = \tilde{\pi}^T(t)\tilde{\Sigma}_y(t)dZ_y(t),\\
&dh(t) = [r_0(t) + \tilde{\pi}(t)^T \tilde{r}_y(t)]dt + \tilde{\pi}^T(t)\tilde{\Sigma}_y(t)dZ_y(t), \qquad h(t_0) = h_0.
\end{aligned}$$

Next, we take $i : \Omega \times T \to \mathbb{R}$ to be the *increase of RWAs before risky asset losses per monetary unit of RWAs*, we denote the volatility in the increase in RWAs before asset losses by σ_i and $Z_i : \Omega \times T \to \mathbb{R}$ represents a standard Brownian motion. Then, we set

$$di(t) = r_i(t)dt + \sigma_i dZ_i(t), \qquad i(t_0) = i_0. \tag{6}$$

The random variable $i(t)$ in (6) may typically originate from RWAs that have recently been accrued or instability in the value of preexisting RWAs that may result from factors such as macroeconomic changes in the bank's loan market.

2.1.3 Stochastic Modeling of Regulatory Capital and RWAs

We can choose from two approaches when modeling our bank in a stochastic setting. The first is a realistic model that incorporates all the aspects of the bank like capital growth, individual accounts and individual clients. Alternatively, we can develop a simple model which acts as a proxy for something more realistic and which emphasizes features that are specific to our particular study. In our situation, we choose the latter option, with stochastic models for the dynamics of bank capital K and RWAs A^r being derived. From our perspective, changes in bank capital K occur when asset losses (see (3) for an exact formulation) are deducted from the total debt and equity raised (see (2) for more details) and the returns on equity (see (5) above). By keeping the above in mind, the dynamics of K may be represented as

$$\begin{aligned} dK(t) = {} & \big[r_0(t)K(t) + K(t)\tilde{\pi}(t)^T\tilde{r}_y(t) + A^r(t)u_1(t) + A^r(t)u_2(t) - A^r(t)r_l(t)\big]dt \\ & + \big[K(t)\tilde{\pi}(t)^T\tilde{\Sigma}_y(t)dZ_y(t) - A^r(t)\sigma_l dZ_l(t)\big]. \end{aligned} \tag{7}$$

On the other hand, small changes in RWAs are modeled by considering the increase in RWAs over asset losses (see (3) for more details) as follows:

$$dA^r(t) = A^r(t)[r_i(t) - r_l(t)]dt + A^r(t)[\sigma_i dZ_i(t) - \sigma_l dZ_l(t)]. \tag{8}$$

2.2 Stochastic System for the Banking Model

The stochastic differential equations (7) and (8) may be rewritten into matrix-vector form in the following way.

Definition 2.1 For $u : \Omega \times T \to \mathbb{R}^{n+1}$ and

$$W(t) = \begin{bmatrix} K(t) \\ A^r(t) \end{bmatrix},$$

define the *stochastic system for the banking model* as

$$dW(t) = E(t)W(t)dt + N(W(t))u(t)dt + a(t)dt + S(W(t), u(t))dZ(t), \tag{9}$$

with the various terms in this SDE being

$$u(t)=\begin{bmatrix} u_2(t) \\ \tilde{\pi}(t) \end{bmatrix}, \qquad E(t)=\begin{bmatrix} r_0(t) & -r_l(t) \\ 0 & r_i(t)-r_l(t) \end{bmatrix},$$

$$N(W(t))=\begin{bmatrix} A^r(t) & K(t)\tilde{r}_y^T(t) \\ 0 & 0 \end{bmatrix}, \qquad a(t)=\begin{bmatrix} A^r(t)u_1(t) \\ 0 \end{bmatrix},$$

$$S(W(t),u(t))=\begin{bmatrix} K(t)\tilde{\pi}(t)^T\tilde{\Sigma}_y(t) & -A^r(t)\sigma_l & 0 \\ 0 & -A^r(t)\sigma_l & A^r(t)\sigma_i \end{bmatrix}, \qquad Z(t)=\begin{bmatrix} Z_y(t) \\ Z_l(t) \\ Z_i(t) \end{bmatrix},$$

where $Z_y(t)$, $Z_l(t)$ and $Z_i(t)$ are mutually (stochastically) independent standard Brownian motions. It is assumed that for all $t \in T$, $\sigma_l(t) > 0$ and $\sigma_i(t) > 0$. Often the time argument of the functions σ_l and σ_i are omitted.

2.3 Capital Adequacy Ratios

In computing the risk-based CAR (compare (1) and the discussion in [18, 19]) we consider the new state variable κ given by (1). It is important for bank solvency that the CAR κ has to maintain a value exceeding some regulatory benchmark ρ. Obviously, low values of κ indicate that the bank is struggling to avoid failure. The next proposition provides an explicit stochastic formula for the dynamics of the CAR κ from (1), that can be verified by means of a straightforward application of Ito's formula.

Proposition 2.1 *Suppose that the dynamics of K and A^r are described by (7) and (8), respectively. Then, the stochastic κ-dynamics of a bank may be represented by the SDE*

$$\begin{aligned} d\kappa(t) &= \kappa(t)\big[r_0(t)+r_l(t)-r_i(t)+\sigma_l^2+\sigma_i^2+\tilde{r}_y(t)^T\tilde{\pi}(t)\big]dt \\ &\quad + \big[u_1(t)+u_2(t)-r_l(t)-\sigma_l^2\big]dt \\ &\quad + \big[\sigma_l^2(1-\kappa(t))^2+\sigma_i^2\kappa(t)^2+\kappa(t)^2\tilde{\pi}(t)^T\tilde{C}(t)\tilde{\pi}(t)\big]^{1/2}d\overline{Z}(t), \\ \kappa(t_0) &= \kappa_0, \end{aligned} \tag{10}$$

where $\overline{Z}: \Omega \times T \to \mathbb{R}$ is a standard Brownian motion.

Note that, in the drift of the SDE (10), the term

$$-r_l(t)+\kappa(t)r_l(t) = -r_l(t)(\kappa(t)-1)$$

appears because it models the effect of asset losses from both the regulatory capital K and RWAs A^r. A similar comment can be made about the term

$$-\sigma_l^2+\kappa(t)\sigma_l^2 = \sigma_l^2(\kappa(t)-1).$$

3 Stochastic Optimization of Capital Adequacy Ratios

In order for a bank owner to determine an optimal rate at which additional debt and equity should be raised and strategy for the allocation of bank equity, it is imperative that a well-defined objective function (loss function in our case) with appropriate constraints is considered. The choice has to be carefully made in order to avoid ambiguous solutions to our stochastic control problem. In this particular paper, we choose to determine a control law $g(t,\kappa(t))$ that minimizes the cost function $J:\mathcal{G}_A\to\mathbb{R}_+$, where $\mathcal{G}_A$ is the class of admissible control laws

$$\mathcal{G}_A=\{g:T\times\mathcal{X}\to\mathcal{U}|g \text{ Borel measurable } \& \ \exists! \text{ solution to closed-loop system}\}, \tag{11}$$

with the closed-loop system for $g\in\mathcal{G}_A$ being given by

$$\begin{aligned} d\kappa(t) &= A(t)\kappa(t)dt+\sum_{j=0}^{n}B_j\kappa(t)g_j(t,\kappa(t))dt+a(t)dt \\ &\quad +\sum_{jj=1}^{3}M_{jj}(g(t,\kappa(t)))\kappa(t)dZ_{jj}(t), \qquad \kappa(t_0)=\kappa_0, \end{aligned} \tag{12}$$

where $M_{jj}(g(t,x(t)))$ is the matrix notation used to denote matrices with entries related to $g(t,x(t))$. In addition, the cost $J:\mathcal{G}_A\to\mathbb{R}_+$, for the banking system in question, may be given by

$$\begin{aligned} J(g) &= \mathbf{E}\bigg[\int_{t_0}^{t_1}\exp(-r_d(s-t_0))b(s,\kappa(s),g(s,\kappa(s)))ds \\ &\quad +\exp(-r_d(t_1-t_0))b_1(\kappa(t_1))\bigg], \end{aligned} \tag{13}$$

where $g\in\mathcal{G}_A$, $T=[t_0,t_1]$, $b_1:\mathcal{X}\to\mathbb{R}_+$ is a Borel measurable function and $r_d\in\mathbb{R}_+$ is the *discount rate*. Furthermore, $b:T\times\mathcal{X}\times\mathcal{U}\to\mathbb{R}_+$ where, for $b_2:\mathcal{U}_2\to\mathbb{R}_+$ and $b_3:\mathbb{R}_+\to\mathbb{R}_+$, we have that

$$b(t,x,u)=b_2(u_2)+b_3(K/A^r).$$

Specific choices for the functions b_1, b_2 and b_3 are made in the ensuing discussion.

3.1 Statement of the Stochastic Optimization Problem

We are now in a position to state the nonlinear optimal stochastic control problem for banks that we solve in a subsequent subsection. The said problem may be formulated as follows.

Problem 3.1 Consider the stochastic control system for banking (12) with the admissible class of control laws, $\mathcal{G}_A\neq\emptyset$, given by (11) and the cost function,

$J:\mathcal{G}_A\to\mathbb{R}_+$, given by (13). Solve $\inf_{g\in\mathcal{G}_A} J(g)$, which amounts to determining the value J^*, $J^*=\inf_{g\in\mathcal{G}_A} J(g)$, and the optimal control law g^*, if it exists, $g^*=\arg\min_{g\in\mathcal{G}_A} J(g)\in\mathcal{G}_A$.

3.2 Solution to the Stochastic Optimization Problem

Consider the CAR stochastic control system (12) for the banking problem with the admissible class of control laws, $\mathcal{G}_A$, given by (11) but with $\mathcal{X}=\mathbb{R}$. In this section, we have to solve

$$J^*=\inf_{g\in\mathcal{G}_A} J(g),$$

$$J(g)=\mathbf{E}\Big[\int_{t_0}^{t_1}\exp(-r_d(s-t_0))\big[b_2(u_2(t))+b_3(\kappa(t))\big]dt + \exp(-r_d(t_1-t_0))b_1(\kappa(t_1))\Big],$$

where $b_1:\mathbb{R}\to\mathbb{R}_+$, $b_2:\mathbb{R}\to\mathbb{R}_+$ and $b_3:\mathbb{R}_+\to\mathbb{R}_+$ are all Borel measurable functions. Next, we state and prove the result.

Theorem 3.1 *Consider the nonlinear stochastic optimization problem for the CAR system* (12) *formulated in Problem* 3.1. *Suppose that the following assumptions hold*:

(i) *The cost function is assumed to satisfy*

$$b_2(u_2)\in C^2(\mathbb{R}),\qquad \lim_{u_2\to-\infty} D_{u_2}b_2(u_2)=-\infty,$$
$$\lim_{u_2\to+\infty} D_{u_2}b_2(u_2)=+\infty,\qquad D_{u_2u_2}b_2(u_2)>0,\quad \forall u_2\in\mathbb{R}.$$

(ii) *There exists a function*, $V:T\times\mathbb{R}\to\mathbb{R}$ *with* $V\in C^{1,2}(T\times\mathcal{X})$, *that is a solution of the partial differential equation* (*PDE*) *of the form*

$$\begin{aligned}0={}&D_tV(t,\kappa)+\frac{1}{2}[\sigma_l^2(1-\kappa)^2+\sigma_i^2\kappa^2]D_{\kappa\kappa}V(t,\kappa)\\&+\kappa\big(r_0(t)+r_l(t)-r_i(t)+\sigma_l^2+\sigma_i^2\big)D_\kappa V(t,\kappa)\\&+[u_1(t)-r_l(t)-\sigma_l^2]D_\kappa V(t,\kappa)+u_2^*(t)D_\kappa V(t,\kappa)\\&+\exp(-r_d(t-t_0))b_2(u_2^*(t))+\exp(-r_d(t-t_0))b_3(\kappa)\\&-\frac{[D_\kappa V(t,\kappa)]^2}{2D_{\kappa\kappa}V(t,\kappa)}\widetilde{r}_y(t)^T\widetilde{C}(t)^{-1}\widetilde{r}_y(t),\end{aligned}\tag{14}$$

where

$$V(t_1,\kappa)=\exp(-r_d(t_1-t_0))b_1(\kappa),\tag{15}$$

and u_2^* *is the unique solution of the equation*

$$0=D_\kappa V(t,\kappa)+\exp(-r_d(t-t_0))D_{u_2}b_2(u_2(t)).\tag{16}$$

Then, the optimal control law is

$$g_2^*(t,\kappa)=u_2^*,\ g_2^*:T\times\mathcal{X}\to\mathbb{R}_+,\quad \textit{with } u_2^*\in\mathcal{U}_2 \textit{ the unique solution of } (16),$$
$$\tilde{\pi}^*=-\frac{D_\kappa V(t,\kappa)}{\kappa D_{\kappa\kappa}V(t,\kappa)}\tilde{C}(t)^{-1}\tilde{r}_y(t),$$
$$g_3^*(t,x)=\min\{1,\max\{0,\tilde{\pi}^*\}\},\ g_3^*:T\times\mathcal{X}\to\mathbb{R}^k.$$

Furthermore, the value of the problem is

$$J^*=J(g^*)=\mathbf{E}[V(t_0,\kappa_0)]. \tag{17}$$

Proof The proof of Theorem 3.1 is contained on the website [31]. □

It is of interest to choose particular cost functions for which an analytic solution can be obtained for the value function and control laws. The following theorem does just that by providing the optimal control laws for a choice of quadratic cost functions.

Theorem 3.2 *Consider the nonlinear optimal stochastic control problem for the CAR system* (12) *formulated in Problem* 3.1 *and the cost function*

$$J(g)=\mathbf{E}\bigg[\int_{t_0}^{t_1}\exp(-r_d(s-t_0))\bigg[\frac{1}{2}c_2(u_2^2(s))+\frac{1}{2}c_3(\kappa(t)-\rho)^2\bigg]ds$$
$$+\frac{1}{2}c_1(\kappa(t_1)-\rho)^2\exp(-r_d(t_1-t_0))\bigg], \tag{18}$$

where $\rho\in\mathbb{R}$ *is some CAR regulatory benchmark. Furthermore, it is assumed that the cost functions satisfy* $b_1(\kappa)=\frac{1}{2}c_1(\kappa-\rho)^2,\ c_1\in(0,\infty);\ b_2(u_2)=\frac{1}{2}c_2u_2^2,\ c_2\in(0,\infty)$ *and* $b_3(\kappa)=\frac{1}{2}c_3(x-\rho)^2,\ c_3\in(0,\infty)$. *Define the ordinary differential equations (ODEs)*

$$-\dot{q}(t)=-q(t)^2/c_2+c_3+q(t)2(r_0(t)+r_l(t)-r_i(t)+\sigma_l^2+\sigma_i^2)$$
$$+q(t)[-r_d-\tilde{r}_y(t)^T\tilde{C}(t)^{-1}\tilde{r}_y(t)+\sigma_l^2+\sigma_i^2],\quad q(t_1)=c_1, \tag{19}$$
$$-\dot{\kappa}_r(t)=-c_3(\kappa_r(t)-\rho)/q(t)-\kappa_r(t)[r_0(t)+r_l(t)-r_i(t)+\sigma_l^2+\sigma_i^2]$$
$$-[u_1(t)-r_l(t)-\sigma_l^2]-(\kappa_r(t)-1)(\sigma_l^2+\sigma_i^2)-\sigma_i^2,\quad \kappa_r(t_1)=\rho, \tag{20}$$
$$-\dot{s}(t)=-r_ds(t)+c_3(\kappa_r(t)-\rho)^2-q(t)\sigma_l^2(\kappa_r(t)-1)^2$$
$$-q(t)\sigma_i^2\kappa_r(t)^2,\quad s(t_1)=0. \tag{21}$$

The function $\kappa_r:T\to\mathbb{R}$ *will be called the* CAR reference process. *In this case, we have that*:

(a) *There exists solutions to the ODEs* (19, 20, 21). *Moreover, for all* $t\in T$, $q(t)>0$.
(b) *The optimal control laws for additional debt and equity and equity allocation are*

$$u_2^*(t)=-(\kappa-\kappa_r(t))q(t)/c_2,\qquad g_2^*(t,\kappa)=u_2^*,\quad g_2^*:T\times\mathcal{X}\to\mathbb{R}_+, \tag{22}$$

and

$$\begin{aligned}&\widetilde{\pi}^*(t) = -\frac{(\kappa-\kappa_r(t))}{\kappa}\tilde{C}(t)^{-1}\widetilde{r}_y(t), \qquad g_3^*: T\times\mathcal{X}\to\mathbb{R}^k,\\ &g_{3,k}^*(t,\kappa) = \begin{cases}\widetilde{\pi}_k^*, & \text{if } \widetilde{\pi}_k^*\in[0,1],\\ \min\{1,\max\{0,\widetilde{\pi}_k^*(t)\}\}, & \text{else},\end{cases} \qquad \forall k\in\mathbb{Z}_n,\end{aligned} \tag{23}$$

respectively.

(c) *The value function and the value of the problem are*

$$V(t,x) = \exp(-r_d(t_1-t_0))\left[\frac{1}{2}(\kappa-\kappa_r(t))^2 q(t) + \frac{1}{2}s(t)\right] \tag{24}$$

and (17), *respectively.*

Proof The proof of Theorem 3.2 is contained on the website [31]. □

4 Illustration of Bank Management Practice under Basel II

In this section, we provide an illustration of some of the features of bank management practice referred to in the above. Our analysis has several connections with the arguments about risk management and regulatory policy in [2, 11] (also [3, 32]). Firstly, we illustrate issues related to credit risk by considering the probability of default of granted loans. The second feature of our example involves the implications of and the interactions between the three pillars of Basel II regulation for bank management that includes a consideration of a regulatory capital constraint. In this regard, we highlight the dynamic interaction between a regulator (who acts in the interest of the public) and bank owner (who, by assumption, acts in the interest of the shareholder). Here, we emphasize that information provided by the bank owner about the level of bank capital K is important for the decision by the supervisor on whether to allow the bank to continue to function or enforce bank closure. Finally, throughout the example, risk incentives, risk shifting and other constraints on banking behavior are referred to. With regard to the latter, realistic constraints associated with the eliciting of additional debt and equity, profitability, incentive compatibility and financing are brought to bear on bank management practice.

4.1 Setting the Scene

Throughout the ensuing illustration, the bank capital K will consist exclusively of equity capital and subordinate debt. Also, the RWAs are solely constituted by loans Λ that carry a risk weight of 100%. Furthermore, we follow a procedure that can be identified with the three-pillared approach of the Basel II capital accord [2]. Pillar 1 (minimum capital requirement) involves the application of a quantitative minimum capital requirement based on public information that determines whether a bank will continue to operate or not. This pillar is related to the likely prompt corrective action that will be taken by supervisors in the event of banks becoming significantly or

Table 1 Categories of banking benchmark regulatory ratios

Categories	ρ		T1CAR		TCAR		TE
Well-capitalized	≥ 0.1	and	≥ 0.06	and	≥ 0.06		–
Adequately capitalized	≥ 0.08	and	≥ 0.04	and	≥ 0.04		–
Undercapitalized	≥ 0.06	and	≥ 0.03	and	≥ 0.03		–
Significantly undercapitalized	< 0.06	or	≥ 0.03	or	≥ 0.03	and	> 0.02
Critically undercapitalized							≤ 0.02

critically undercapitalized. In this regard, Table 1 below makes a distinction between the capitalization states of banks with respect to several benchmark regulatory ratios.

In Table 1, we have that TCAR, T1CAR and TE are the abbreviations for the total CAR (also known as the leverage ratio), Tier 1 CAR and tangible equity, respectively. Here the TCAR and T1CAR is the regulatory capital-to-total assets ratio (see discussion in Sect. 5.1.3) and T1 capital-to-total assets ratio, respectively. As is the case for the TCAR and T1CAR, the CAR ρ in the first column of Table 1 gives an indication (in terms of the level of capitalization of the bank) of significant values for the benchmark CAR. In Pillar 2 (supervisory review), the bank decides on whether the bank capital held is sufficient to invest in a certain credit risk-type or whether it is necessary to elicit capital by issuing debt and raising equity. At this stage, we distinguish between failed, capital-constrained and capital-unconstrained banks. Pillar 3 (market discipline) offers the supervisor another opportunity to terminate banking operations based on disclosure about the posterior probability of failure of the bank ([32] for more details).

4.2 Pillar 1—Minimum Capital Requirement

Most banks consider the level of capital to be the binding constraint in deciding on whether to issue a loan or not. In order for a loan to be granted, the risk adjusted rate of return on a particular loan must exceed the return on capital. The Basel II capital accord contains the *total capital constraint* that, in our case, relates RWAs A^r to capital K via the inequality

$$K(t) \geq \rho A^r(t), \quad \text{or equivalently,} \quad \kappa(t) \geq \rho, \tag{25}$$

where ρ denotes a CAR regulatory benchmark. The setting of a regulatory benchmark is an attempt to encourage banks to hold a RBC-to-RWA ratio, ρ, of at least 8% [10, 15, 29]. The cost on meeting the obligations related to the aforementioned capital constraint (25) is encoded in a cost on the CAR (see (18) in Theorem 3.2 for the exact form of the cost function). If $\kappa < \rho$, then there should be a strictly positive cost. On the other hand, if $\kappa > \rho$, then there may be a cost though most banks will be satisfied and not impose such a cost. In Theorem 3.2, we have selected the cost function

$$b_3(\kappa) = \frac{1}{2}c_3(\kappa - \rho)^2.$$

This is done also to obtain an analytic solution of the value function and that case by itself is interesting. Several approaches to the management of bank closure and its relationship with the minimum capital requirement (Pillar 1) described in Basel II exist. In the event of closure, the loans to private agents, Λ, are liquidated at a cost of $\lambda\Lambda$, where λ is determined exogenously. The implementation of Pillar 1 via the liquidation cost approach will mainly be impacted by the level of bank capital and the CAR. In this regard, the supervisor has to decide whether capital, K, as described by (7), covers the cost of liquidation, λL. If we have, for $t \in [0, t_1)$, that

$$K(t) \geq \lambda\Lambda(t), \quad \text{or equivalently,} \quad \kappa(t) = \frac{K(t)}{\Lambda(t)} \geq \lambda, \tag{26}$$

then the bank exceeds the minimum capital requirement and is able to continue operating. If condition (26) fails, closure may occur since it is unlikely that the bank will be able to re-capitalize itself.

4.3 Pillar 2—Supervisory Review

In this subsection, we consider the interaction between the capital adequacy management of a bank with respect to its fundamental function of granting loans and supervisory constraints on, for instance, eliciting debt and equity (see [32] for more details). Having exceeded a minimum regulatory capital requirement in the first stage, under supervisory constraint, the bank may now acquire a new credit risk type, Λ_1. The return on Λ_1 is reliant on whether the profit Π is

$$\text{either} \quad \Pi(t) = \Pi_p(t) \geq 0 \quad \text{or} \quad \Pi(t) = \Pi_n(t) < 0. \tag{27}$$

The *social value* of acquisition Λ_1 is $(1-\psi)\Pi_p(t) - \psi\Pi_n(t)$, where ψ is the *anterior probability of bank failure*. In this regard, Π_p and Π_n from (27) correspond to the favored (low ψ) and the unfavored (high ψ) bank operational states, respectively. Here we suppose that the bank has no direct costs associated with failure ([17] for more details). Next, the bank assesses whether its level of capital is high enough to invest in Λ_1 by determining whether the Λ_1-capital constraint

$$\kappa(t) = \frac{K(t)}{\Lambda_1(t)} \geq 1 \tag{28}$$

is satisfied (compare (28) to the total capital constraint (25) described above). If inequality (28) holds, there is no need to elicit additional debt or equity. If, on the other hand, we have

$$\frac{\lambda\Lambda(t)}{\Lambda_1(t)} \leq K(t) < 1,$$

the bank has to acquire additional financing from, for instance, debt and shareholders. In this regard, the market may impose certain restrictions on the amount of debt and equity that the bank can raise. This *eliciting constraint* is intended to discourage the bank from investing in riskier assets which have a higher default probability. For sake of argument, we suppose that Λ_1 is replaced by another credit-risk type, $\widetilde{\Lambda}_1$, with a higher return $\widetilde{\Pi}_p(t) > \Pi_p(t)$ and a higher probability of failure $\widetilde{\psi} > \psi$. Let K^* be the additional debt and equity elicited with G being defined by

$$\text{Gross Return to Share- and Debt-holders} = \begin{cases} G, & \text{in favored operational state,} \\ 0, & \text{otherwise.} \end{cases}$$

Next, we introduce two constraints that is commensurate with prudent bank management practice. In order to make $\widetilde{\Lambda}_1$ less attractive in the *absence* of elicited debt and equity, we require that the returns on Λ_1 and $\widetilde{\Lambda}_1$ satisfy the *profitability constraint* given by

$$(1-\psi)\Pi_p(t) \geq (1-\widetilde{\psi})\widetilde{\Pi}_p(t).$$

On the other hand, to discourage the shifting of risk in the *presence* of debt and equity, $G > 0$ must satisfy the *incentive compatibility constraint* expressed as

$$0 < G(t) \leq \Pi_p(t) - \frac{1-\widetilde{\psi}}{\widetilde{\psi}-\psi}\Big[\widetilde{\Pi}_p(t) - \Pi_p(t)\Big]. \tag{29}$$

The aforementioned debt and equity satisfy *individual rationality* if, at equilibrium, it guarantees an outcome such that the profit for the debt- and shareholders exceeds a certain level. This concept is useful when the debt- and shareholders have the option to terminate their involvement. In our case, such rationality leads to

$$(1-\psi)G(t) \geq \beta K^*(t), \tag{30}$$

where the market requirement $\beta > 1$ is the gross return on capital. Inequalities (29, 30) together suggest that the maximum additional amount of debt and equity, $\widetilde{K}^*$, the bank may raise, may be expressed as

$$K^*(t) \leq \frac{1-\psi}{\beta}\left[\Pi_p(t) - \frac{1-\widetilde{\psi}}{\widetilde{\psi}-\psi}\left\{\widetilde{\Pi}_p(t) - \Pi_p(t)\right\}\right] := \widetilde{K}^*(t).$$

In essence, Stage 2 distinguishes between failed, capital-constrained and capital-unconstrained bank types whose classification depends on how their level of capital compares with regulatory benchmarks (see Table 1).

4.3.1 Failed Bank

A bank for whom the capital adequacy ratio K, induced by Λ_1, is subject to the condition

$$K(t) < 1 - \frac{\widetilde{K}^*(t)}{\Lambda_1(t)}$$

cannot raise enough debt and equity to invest in Λ_1 and fails. If this happens, bank owners receive the market value of $K(t) - \lambda\Lambda(t)$ which is positive because the bank exceeded the minimum capital requirement from the first pillar. The supervisor has to cover certain costs when a bank fails. For instance, if upon failure, the resulting operational state is favorable, the opportunity costs of total profits that have been lost in the liquidation process is charged to the supervisor but credit is given for the *net social value at closure*

$$\Lambda_1(t) - \lambda\Lambda(t). \tag{31}$$

The supervisor may also incur a deadweight loss, l_{dw}, where, for instance, external income with positive returns are foregone when a favored bank is closed or legal disputes arise from a bank that was deemed to be viable after closure.

4.3.2 Capital-Constrained Bank

The bank, that has insufficient capital K to invest in Λ_1, may face a *financing constraint*

$$1 > K(t) \geq 1 - \frac{K^*(t)}{\Lambda_1(t)}$$

and may be subject to an implicit capital requirement from the market of the form

$$K(t) \geq 1 - \frac{K^*(t)}{\Lambda_1(t)} \geq 1 - \frac{\widetilde{K}^*(t)}{\Lambda_1(t)}.$$

Such a bank satisfies the minimum market capital requirement, but is capital-constrained and must issue debt and raise equity to grant Λ_1 in loans. If the constrained bank is allowed to operate and it fails, the supervisor must deal with both the operating loss Π_n (which the bank's owners do not bear because of assumed limited liability) and the cost of liquidation, $\lambda\Lambda$. Compared with (31), if the bank fails it follows that the net social value at closure is given by

$$K(t) + K^*(t) - \lambda\Lambda(t)$$

and the loss is $-\Pi_n - \Lambda_1(t)$. In the constrained case, the total social investment including debt and equity is Λ_1.

4.3.3 Capital-Unconstrained Bank

Finally, a bank with a Λ_1-induced CAR K, that satisfies

$$K(t) \geq 1, \tag{32}$$

is unconstrained and may invest in Λ_1 without raising additional debt and equity. In that case, the excess bank capital

$$K(t) - \Lambda_1(t) \tag{33}$$

may be invested in a riskless asset (such as treasuries) that provides a zero net return. If the bank is unconstrained, the *aggregate social value* is always more than the constrained case by the amount of excess capital given by (33). However, if the unconstrained bank invests any excess capital in a riskless asset with zero net return then all consequences are exactly larger by (33).

4.4 Pillar 3—Market Discipline

Pillar 3 aims to strengthen market discipline by insisting on enhanced disclosure by banks. The *disclosure requirements* will enable market roleplayers to access information about the bank's capital adequacy and risk exposures. If the bank successfully complies with the conditions above, in the third stage the bank acquires information that is relevant to its continued operational management. This information informs the ultimate regulatory decision on whether to take corrective action or enforce bank closure ([32] for more details). The newly acquired information may be in the form of a signal

$$I = \begin{cases} Y, \\ N. \end{cases} \tag{34}$$

This signal depends on the observed value of I and results in a *posterior probability of bank failure* that may be expressed as $p_i = P(\text{failure} \mid I = i),\ \ i = Y$ or N. Depending on its incentives, the bank discloses the correct or incorrect value of I to the supervisor. Since by definition

$$\psi = p_Y P(I = Y) + p_N P(I = N),$$

we have that $p_Y \leq \psi \leq p_N$. Our illustrative example of bank management practice is concluded by briefly mentioning the role that the *information signal* I in (34) can play in the interaction between the supervisor and bank owner. Before a decision about corrective action or bank closure is made, the supervisor requests the value of I from the bank owner. If the supervisor's decision rule is incentive-compatible it must not impose a penalty on the disclosure of the authentic value of I.

5 Economic Analysis of the Main Issues

In accordance with the dictates of the Basel II capital accord, the models of bank items constructed in this paper are related to the methods currently being used to assess the riskiness of bank portfolios and their minimum capital requirement [8, 10].

5.1 Banking Model

In this subsection, we analyze aspects of the bank model presented in Sect. 2.

5.1.1 Regulatory Bank Capital and RWAs

Despite the analysis in Sect. 2.1, bank capital is notoriously difficult to define monitor and measure. With regard to the latter, the measurement of equity depends on how all of a bank's financial instruments and other assets are valued. The description of the shareholder equity component of bank capital y is largely motivated by the following two observations. Firstly, it is meant to reflect the nature of the book value of equity.

Our intention is also to recognize that the book and market value of equity is highly correlated.

Under Basel II, bank capital requirements have replaced reserve requirements as the main constraint on the behavior of banks [33]. A first motivation for this is that bank capital has a major role to play in overcoming the moral hazard problem arising from asymmetric information between banks, creditors and debtors. Also, bank regulators require capital to be held to protect themselves against the costs of financial distress, agency problems and the reduction of market discipline caused by the safety net.

5.1.2 Stochastic System for the Banking Model

From the stochastic system given by (9) it is clear that $u = (u_2, \widetilde{\pi})$ affects only the SDE of $K(t)$ but not that of $A^r(t)$. In particular, u_2 affects only the drift of $K(t)$. On the other hand, for (9), we have that $\widetilde{\pi}$ affects the variance of $K(t)$ and the drift of $K(t)$ via the term $K(t)\widetilde{r}_y(t)^T\widetilde{\pi}(t)$.

5.1.3 Capital Adequacy Ratios

It is an accepted fact that cyclicality is at the root of financial instability in the banking industry. In this regard, under Basel II, capital requirements are likely to increase in recessions. Yet if capital requirements show this tendency—when building reserves from decreasing profits is difficult or raising fresh capital is likely to be extremely costly—banks would have to reduce their loans and the subsequent credit crunch would add to the downturn. This would make the recession deeper, thus setting in motion an undesirable vicious circle that might ultimately have an adverse effect on the stability of the banking system. This is why capital requirements are said to be procyclical despite actually increasing (decreasing) during a downturn (upturn). The implications of this link between financial stability and macroeconomic stability in terms of the soundness of credit banks merit being taken into account in the final design of Basel II.

5.2 Stochastic Optimization of Capital Adequacy Ratios

In this section, we analyze total bank capital, binding capital constraints, retained earnings and bank value for a shareholder.

5.2.1 Statement of the Stochastic Optimization Problem

In this subsubsection, we discuss the cost function (see (13, 18) for an exact formulation) arising from the discussion in Sect. 3.1.

As far as the cost function is concerned, the control objectives are to meet the bank's regulatory obligations and to elicit from the debt- and shareholders as little additional debt and equity as possible. The first control objective is formulated as a cost on the CAR κ, while the second is formulated as a cost on the additional debt and equity raised, u_2. As to the mathematical form for the cost functions, we have considered several options, discussed below. Of course, one can formulate any cost function. The question then is whether the resulting dynamic programming equation can be solved analytically? We have obtained an analytic solution so far only for the case of quadratic cost functions (see Theorem 3.2 for an exact formulation). For the cost on the rate at which additional debt and equity is raised by debt- and shareholders, the function $b_2(u_2)$ is considered. If $u_2 > 0$ then additional debt and equity need to be raised in order to guarantee the safety of the bank. The cost function should be such that extra asset losses are penalized, hence $u_2 > 0$ should imply that $b_2(u_2) > 0$. The situation where amounts are paid back to the debt- and shareholders, $u_2 < 0$, is of course possible and may take the form of dividends (compare with [34] for insurance). In this regard, it may also be good to retain a temporary overcapitalization for dividends payments to debt- and shareholders at a later stage or for periods of undercapitalization. In general it seems best not to penalize payouts to debt- and shareholders. Another option is to restrict the input variable u_2 to the set $\mathbb{R}_+$. In Theorem 3.2, we have selected the cost function $b_2(u_2) = \frac{1}{2}c_2u_2^2$. This penalizes both positive and negative values of u_2 in equal ways. The only reason for making this choice is that an analytic solution of the value function can be obtained.

5.2.2 *Solution to the Stochastic Optimization Problem*

In this subsubsection, we discuss the optimal control law (see (22, 23) for more details) that was discussed in Theorem 3.2 of Sect. 3.2. From this theorem, we have that the formulas for the optimal control law are

$$u_2^* = g_2^*(t,\kappa) = -(\kappa - \kappa_r(t))q(t)/c_2, \tag{35}$$

$$\widetilde{\pi}^* = g_1^*(\kappa) = -\frac{\kappa - \kappa_r(t)}{\kappa}\widetilde{C}(t)^{-1}\widetilde{r}_y(t). \tag{36}$$

An interpretation of this control law follows. The optimal rate at which the additional debt and equity is elicited, u_2^*, is proportional to the difference between the CAR κ and the reference process for this ratio, $\kappa_r(t)$, at time $t \in T$ (see (35) for more details). The proportionality factor, $q(t)/c_2$, depends on the relative ratio of the cost function on u_2 and on the deviation of the CAR from the reference ratio, $\kappa - \kappa_r$. The property that the control law is symmetric in κ with respect to the reference process κ_r is a consequence of the cost function $b_3(\kappa) = \frac{1}{2}c_3(\kappa - \kappa_r)^2$ being symmetric with respect to $\kappa - \kappa_r$. The optimal equity distribution (see (36) for a precise mathematical description) is proportional to the relative difference between the CAR and its reference process, $(\kappa - \kappa_r(t))/\kappa$. In this case, the proportionality factor is $\widetilde{C}(t)^{-1}\widetilde{r}_y(t)$ which represents the relative rates of increase of bank equity multiplied with the inverse of the corresponding variances. It is surprising that the control law has this structure. Apparently the optimal control law is not to sell first all equity with the highest variance or the lower rate, then equity with the next highest variance or the next to lowest rate, etc. The proportion of all equity depends on the relative weighting in $\widetilde{C}(t)^{-1}\widetilde{r}_y(t)$ and not on the deviation $(\kappa - \kappa_r(t))$. The novel structure of the optimal control law is the *reference process for the CAR* $\kappa_r : T \to \mathbb{R}$. The ODE for this process is

$$\begin{aligned}
-\dot{\kappa_r}(t) &= -c_3(\kappa_r-\rho)/q(t) - \kappa_r(t)\big[r_0(t)+r_l(t)-r_i(t)+\sigma_l^2+\sigma_i^2\big] \\
&\quad - \big[u_1(t)-r_l(t)-\sigma_l^2\big] - (\kappa_r(t)-1)(\sigma_l^2+\sigma_i^2)-\sigma_i^2, \\
\kappa_r(t_1) &= \rho.
\end{aligned} \tag{37}$$

This ODE is new for the area of banking regulation and therefore deserves further discussion. Equation (37) has several terms on its right-hand side which will be discussed separately. Consider the term $u_1(t)-r_l(t)-\sigma_l^2$. This represents the difference between primary inflow and outflow of the regulatory capital and RWAs. Note that $u_1(t)$ is the normal rate at which debt- and shareholder contributions are elicited and $r_l(t)$ is the rate of the decrease of bank capital due to asset losses. Note that if $u_1(t)-r_l(t)-\sigma_l^2>0$ then the CAR reference process κ_r can be increasing in time due to this inequality so that, for $t>t_1$, $\kappa(t)<\rho$. The term $c_3(\kappa_r(t)-\rho)/q(t)$ models that if the CAR reference process κ_r is smaller than ρ then the function has to increase with time. The quotient $c_3/q(t)$ is a weighting term which accounts for the running costs and for the effect of the solution of the Riccati differential equation (19). The term

$$\kappa_r(t)\big[r_0(t)+r_l(t)-r_i(t)+\sigma_l^2+\sigma_i^2\big]$$

accounts for two effects. The difference $r_l(t)-r_i(t)$ is the net effect of the rate of asset losses, r_l, and that of the increase in RWAs. The term $r_0(t)+\sigma_l^2+\sigma_i^2$ is the effect of the increase in the RWAs due to subordinate debt and the variance of the bank equity. The last term, $(\kappa_r(t)-1)(\sigma_l^2+\sigma_i^2)-\sigma_i^2$, accounts for the fact that asset losses have an effect on both the regulatory capital and RWAs.

More information about (37) is obtained by streamlining the ODE for κ_r. Assume that the parameters of the problem are all time-invariant and also that q has become constant with value q_0. Then, the differential equation for κ_r can be rewritten as

$$-\dot{x}_r(t) = -k(\kappa_r(t)-m), \quad \kappa_r(t_1)=\rho;\ k=(r_0+r_l-r_1+2(\sigma_l^2+\sigma_i^2))+c_3/q_0,$$

$$\xi = \frac{\rho c_3/q_0-(u_1-r_l-\sigma_l^2)+\sigma_l^2}{(r_0+r_l-r_i+2(\sigma_l^2+\sigma_i^2))+c_3/q_0}.$$

Because the finite horizon is an artificial phenomenon to make the optimal stochastic control problem tractable, it is of interest to consider the long term behavior of the CAR reference process κ_r. If the values of the parameters are such that $k>0$ then the differential equation with the terminal condition is stable. If this condition holds then $\lim_{t\downarrow 0} q(t)=q_0$ and $\lim_{t\downarrow 0}\kappa_r(t)=\xi$, where $\downarrow$ prescribes to start at t_1 and to let t decrease to 0. Depending on the value of ξ, the control law for $t_1-t\to\infty$ becomes

$$u_2^*(t) = -(\kappa(t)-\xi)q_0/c_2 = \begin{cases} >0, & \text{if } \kappa(t)<\xi, \\ <0, & \text{if } \kappa(t)>\xi, \end{cases}$$

$$\pi^*(t) = -\frac{(\kappa(t)-\xi)}{\kappa(t)}\widetilde{Cr}_j = \begin{cases} >0, & \text{if } \kappa(t)<\xi, \\ <0, & \text{if } \kappa(t)>\xi, \end{cases} \quad \text{if } \pi^*<0 \ \text{ then set } \pi^*=0.$$

The interpretation for the two cases are as follows:

Case 1: ($\kappa(t) > \xi$). In this case, the CAR κ is too high. This situation is penalized by the cost function and as a consequence, the control law prescribes not to contribute bank equity and to pay higher dividends due to overcapitalization to the debt- and shareholders (compare the analogue for insurance in [34]). The dividend advice is due to the quadratic cost function which was selected to make the solution analytically tractable.

Case 2: ($\kappa(t) < \xi$). In this case, the CAR κ is too low. As in the previous case, the cost function penalizes and the control law implies that there should be more investment in bank equity and that additional debt should be elicited. Both of the aforementioned scenarios will lead to a higher CAR in the long run.

An expression for the difference $\xi - \rho$ is not obvious and depends on many parameters of the problem as it should. However, our analysis intimates that, for a fixed value of the CAR regulatory benchmark, ρ, and cost function for the CAR, the value of ξ is strictly less than that of ρ. For a fixed value of ρ, the value of ξ decreases if the rate of return from subordinate debt, r_0, increases. Accordingly, the CAR which needs to be held in focus for a long horizon, can be somewhat less than the value ρ of the cost function. Correspondingly, if the bank owners use a control law as above with ρ instead of ξ then the fund will generate a higher CAR κ than required. Of course, this conclusion is based on all the assumptions which have been used in the model.

5.3 Illustration of Bank Management Practice under Basel II

In this subsection, we provide some comments about the illustrative example in Sect. 4.

5.3.1 Setting the Scene

The illustration in Sect. 4 mainly deals with capital requirements but is also loosely related to asset requirements that are formulated by the bank's shareholders and regulators. In this regard, the value of the bank's asset portfolio A is allowed to evolve without any restriction on time until it becomes less than a critical asset value α^s that is chosen by the shareholders and initiates the default process. In addition, we can consider a related prescribed asset value α^r set by the regulator, that is instrumental in determining a threshold for bank closure and reorganization. The problem of determining and characterizing α^s and α^r and their interrelationship is sometimes called the *asset threshold problem*.

5.3.2 Pillar 1—Minimum Capital Requirements

Pillar 1 of Basel II intends to provide a stronger link between the management of capital requirements and actual risk.

5.3.3 Pillar 2—Supervisory Review

Pillar 2 focuses on strengthening the supervisory process, particularly in assessing the quality of risk management in banking institutions and in evaluating whether these institutions have adequate procedures to determine how much capital they need.

5.3.4 Pillar 3—Market Discipline

Pillar 3 involves the improvement of market discipline through increased disclosure of details about the bank's credit exposures, its amount of reserves and capital, the bank owners and the effectiveness of its internal ratings system. Since bank management has become increasingly complicated and supervisors (acting as representatives of the depositors' interests) battle to monitor banking activities, the recourse to market discipline appears to be justified. In this regard, monitoring of banks by professional investors and financial analysts as a complement to banking supervision is being encouraged. However, the manner in which market discipline and the other two pillars are to be managed in concert with each other is a subject of much debate.

6 Conclusions

In this paper, Theorem 3.2 established the level of returns on bank capital and the rate of acquisition of additional debt and equity that is needed to attain an optimal CAR level. In addition, we commented on an obvious shortcoming of Basel II regulation by introducing the notion of a CAR reference process κ_r in Sect. 3.2. In Sect. 4, we considered some of the management practice issues that relate to our stochastic dynamic models of bank behavior.

The main thrust of future research will involve models of bank items driven by Lévy processes [35, Chap. I, Sect. 4]. Such processes have an advantage over the more traditional modelling tools such as Brownian motion in that they describe the non-continuous evolution of the value of economic and financial items more accurately. For instance, because the behavior of bank loans, wealth, capital and CARs are characterized by jumps, the representation of the dynamics of these items by means of Lévy processes is more realistic. As a result of this, recent research has strived to replace the existing Brownian motion-based bank models [2, 11, 16, 19, 26, 27] by systems driven by more general processes. Also, a study of the optimal capital structure should ideally involve the consideration of taxes and costs of financial distress, transformation costs, asymmetric bank information and the regulatory safety net. Another research area that is of ongoing interest is the (credit, market, operational, liquidity) risk minimization of bank operations within a regulatory framework [22, 30]. Another risk that becomes important is *interest rate risk* at the point of loan issuing. For instance, an alternative optimization problem would be to maximize the risk-free rate of interest in order to provide a shareholder with an incentive to invest money.

References

1. Dangl, J.-P., Lehar, B.: Value-at-risk vs. building block regulation in banking. J. Financ. Intermediation **13**, 96–131 (2004)
2. Decamps, J.-P., Rochet, J.-C., Roger, B.: The three pillars of Basel II: optimizing the mix. J. Financ. Intermediation **13**, 132–155 (2004)
3. Diamond, D.W., Rajan, R.G.: A theory of bank capital. J. Finance **55**, 2431–2465 (2000)
4. Basel Committee on Banking Supervision: International convergence of capital measurement and capital standards. Bank for International Settlements (July 1988). http://www.bis.org/publ/bcbs04A.pdf
5. Basel Committee on Banking Supervision: Amendment to the capital accord to incorporate market risks. no. 24. Bank for International Settlements (January 1996). http://www.bis.org/publ/bcbs04A.pdf
6. Basel Committee on Banking Supervision: Amendment to the capital accord to incorporate market risks. Bank for International Settlements (September 1997)
7. Basel Committee on Banking Supervision: First consultative paper. Bank for International Settlements (June 1999). http://www.bis.org/bcbs/bcbscp1.htm
8. Basel Committee on Banking Supervision: Second consultative paper. Bank for International Settlements (January 2001). http://www.bis.org/bcbs/bcbscp2.htm
9. Basel Committee on Banking Supervision: The new Basel capital accord. Bank for International Settlements (January 2001). http://www.bis.org/publ/bcbsca.htm
10. Basel Committee on Banking Supervision: International convergence of capital measurement and capital standards: a revised framework. Bank for International Settlements (June 2004). http://www.bis.org/publ/bcbs107.pdf
11. Rochet, R.: Rebalancing the three pillars of Basel II. Econ. Policy Rev. **10**, 7–25 (2004)
12. Basel Committee on Banking Supervision: Credit risk modelling: current practices and applications. Bank for International Settlements (1999)
13. Basel Committee on Banking Supervision: Regulatory treatment of operational risk. Working Paper, Bank for International Settlements (September 2001). http://www.bis.org/publ/bcbsca.htm
14. Modigliani, F., Miller, M.: The cost of capital, corporation finance and the theory of investment. Am. Econ. Rev. **48**, 261–297 (1958)
15. Berger, A.N., Herring, R.J., Szegö, G.: The role of capital in financial banks. J. Bank. Finance **19**, 393–430 (1995)
16. Leland, H.: Risky debt, bond covenants and optimal capital structure. J. Finance **49**, 1213–1252 (1994)
17. Rochet, R.: Capital requirements and the behaviour of commercial banks. Eur. Econ. Rev. **36**, 1137–1178 (1992)
18. Mukuddem-Petersen, J., Petersen, M.A.: Stochastic behaviour of risk-weighted bank assets under Basel II. Appl. Financ. Econ. Lett. **1**, 133–138 (2005)
19. Fouche, C.H., Mukuddem-Petersen, J., Petersen, M.A.: Continuous-time stochastic modeling of capital adequacy ratios for banks. Appl. Stoch. Models Business Ind. **22**, 41–71 (2006)
20. Repullo, R.: Capital requirements, market power, and risk-taking in banking. J. Financ. Intermediation **13**, 156–182 (2004)
21. Blum, J.: Do capital adequacy requirements reduce risks in banking? J. Bank. Finance **23**, 755–771 (1999)
22. Mukuddem-Petersen, J., Petersen, M.A.: Bank management via stochastic optimal control. Automatica **42**, 1395–1406 (2006)
23. Pellizzon, L., Schaefer, S.: Pillar 1 vs pillar 2 under risk management. In: Carey, M., Stulz, R. (eds.) NBER Conference on Risks of Financial Institutions, October 2004
24. Basak, S., Shapiro, A.: Value-at-risk based management: optimal policies and asset prices. Rev. Financ. Stud. **14**, 371–405 (2001)
25. Cuoco, D., Liu, H.: An analysis of VAR-based capital requirements. J. Financ. Intermediation **15**, 362–394 (2006)
26. Gideon, F., Mukuddem-Petersen, J., Petersen, M.A.: Minimizing banking risk in a Lévy process setting. J. Appl. Math. (2007). doi:10.1155/2007/32824
27. Mukuddem-Petersen, J., Petersen, M.A., Schoeman, I.M., Tau, B.A.: An application of stochastic optimization theory to institutional finance. Appl. Math. Sci. **1**, 1357–1384 (2007)
28. Mukuddem-Petersen, J., Petersen, M.A., Schoeman, I.M., Tau, B.A.: Maximizing banking profit on a random time interval. J. Appl. Math. (2007). doi:10.1155/2007/29343

29. Dewatripont, M., Tirole, I.: The Prudential Regulation of Banks. MIT Press, Cambridge (1994)
30. Jackson, P., Perraudin, W.: Regulatory implications of credit risk modelling. J. Bank. Finance **24**, 1–14 (2000)
31. North-West University (Potchefstroom Campus): http://www.puk.ac.za/navorsing/eenheidbwi/mense (November 2007)
32. Estrella, A.: Bank capital and risk: is voluntary disclosure enough? J. Financ. Serv. Res. **26**, 145–160 (2004)
33. Bosch, T., Mukuddem-Petersen, J., Petersen, M.A., Schoeman, I.M.: Optimal auditing in the banking industry. Optim. Control Appl. Methods (2007). doi:10.1002/oca.828
34. Petersen, M.A., Raubenheimer, H., Van Der Walt, M.: Stochastic approach to dividend equalization fund modeling and solvency. Math. Models Methods Appl. Sci. **15**, 1795–1810 (2005)
35. Protter, P.: Stochastic Integration and Differential Equations, 2nd edn. Springer, Berlin (2004)